The Bisection Method and Newton's Method.

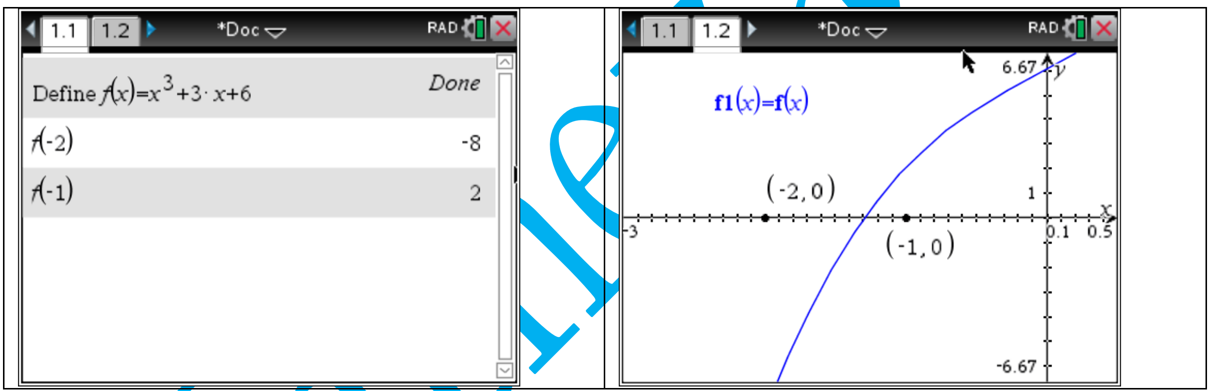
If $f(x)$ a function, then a number r for which $f(r)=0$ is called a **zero** or a **root** of the function $f(x)$, or a **solution** to the equation $f(x)=0$. You are already familiar with ways of solving linear and quadratic equations. However, not all equations admit solutions that can be found exactly. There are methods of finding approximate solutions to equations, one of which is the bisection method, which depends on the intermediate value theorem.

In simple words the theorem states that if a function $f(x)$ is continuous on the interval (a,b) and has one solution to the equation $f(x)=0$ in the interval $[a,b]$ then the function has to have opposite signs at the endpoints of this interval. In other words the sign of $f(a)$ has to be opposite to the sign $f(b)$.

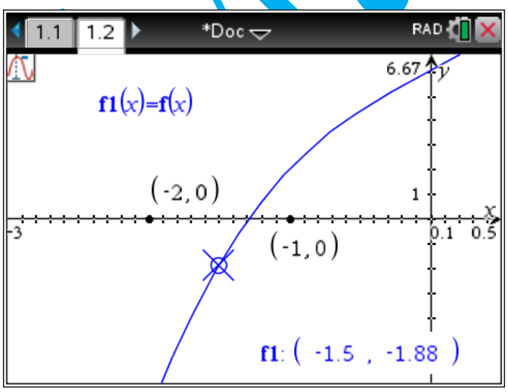
The **bisection method** uses this fact and successive approximations to locate the zero as shown in the following exercise.

Exercise 1

Consider the function $f(x)=x^3+3x+6$ with a zero in the interval $[-2,-1]$.
Evaluating $f(-2)=-8<0$ and $f(-1)=2>0$ confirms the root lies in this interval as the cubic function is continuous over R.

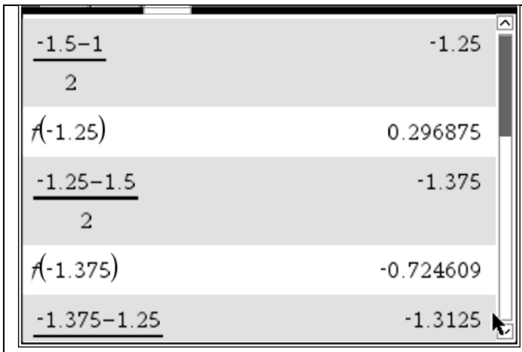


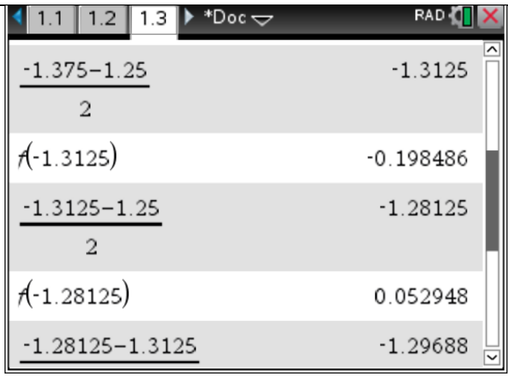
Calculate the midpoint of the interval $[-2,-1]$ as $\dfrac{-2-1}{2}=-1.5$ and evaluate $f(-1.5)$.

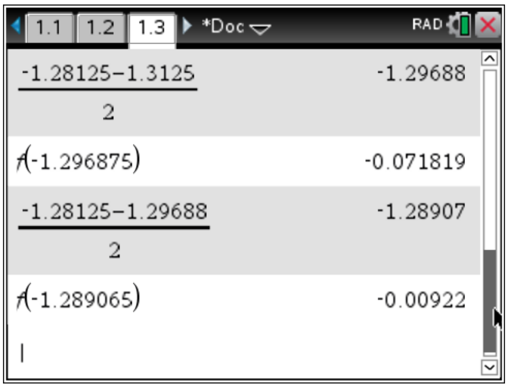


As $f(-1.5)=-1.88<0$ the root lies in the interval [-1.5,-1].

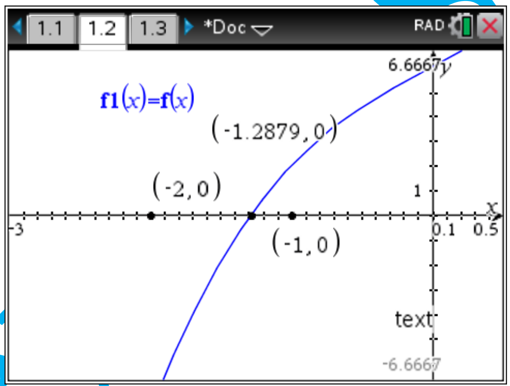
Continue the iterations until you obtain the value of the root correct to 2 decimal places.





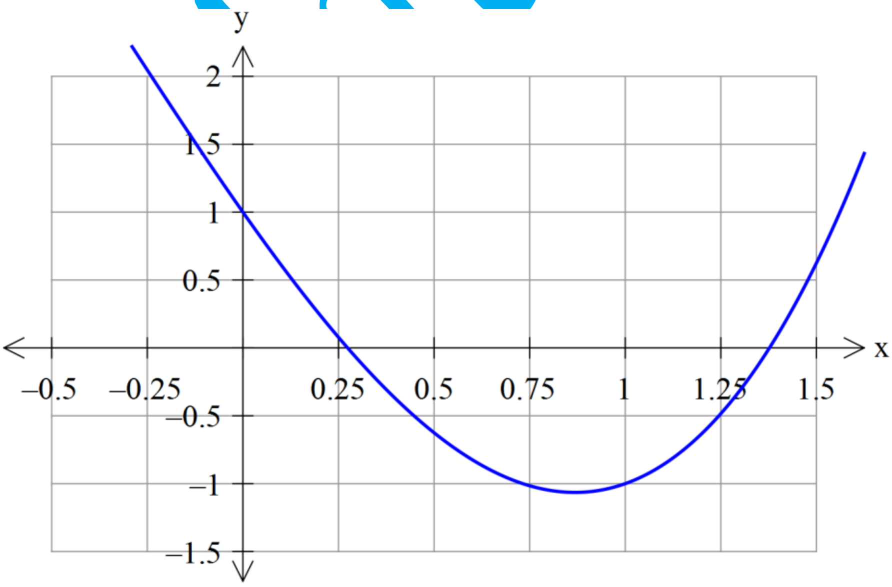


The approximate value of the root is -1.29 correct to two decimal places.



Question 1

Consider the function $f(x) = x^3 + x^2 - 4x + 1$, whose graph is given below.



(a) Evaluate $f(0)$ and $f(0.5)$.

(b) Taking the midpoint of $[0, 0.5]$ as 0.25 evaluate $f(0.25)$.

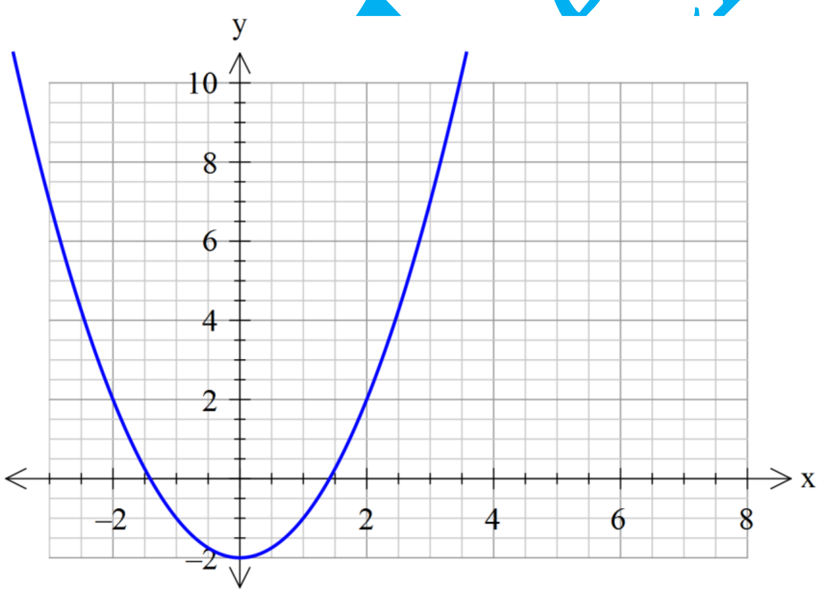
(c) Decide whether to choose the next midpoint in the interval $[0, 0.25]$ or $[0.25, 0.5]$.

(d) Continue this process until you get a root answer correct to two decimal places.

Newton's Method.

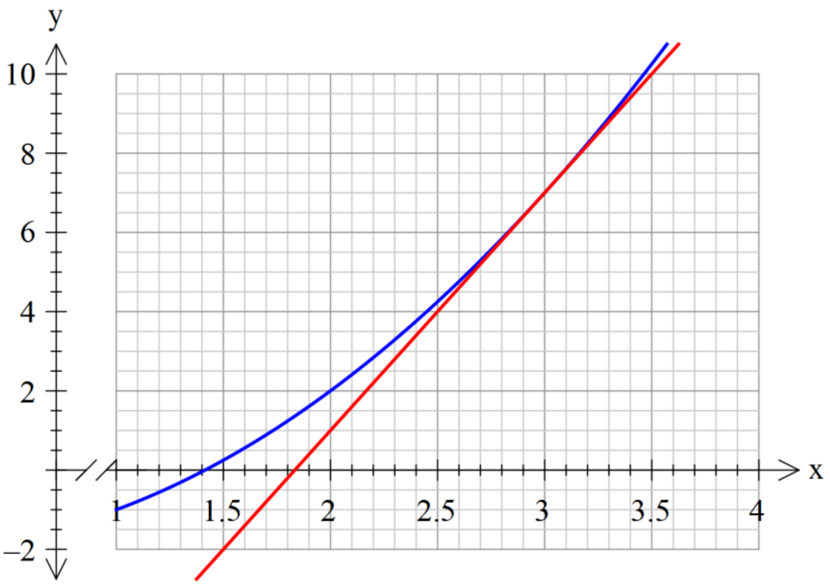
All iterative methods require a first approximation to a root. By sketching the graph we can form an idea of where the function will intersect the *x*-axis.

For example, consider the graph of $f(x) = x^2 - 2$, which you should be able to sketch by hand.



Suppose we take the first approximation to the positive root to be $x = 3$.

By enlarging the region around this root, it can be seen that a better approximation can be found where the tangent at $x = 3$ cuts the *x*-axis.



We can find the equation of the tangent as follows

$$f'(x) = 2x$$
$$y - f(3) = f'(3)(x-3)$$
$$y - 7 = 6(x-3)$$
$$y = 6x - 11$$

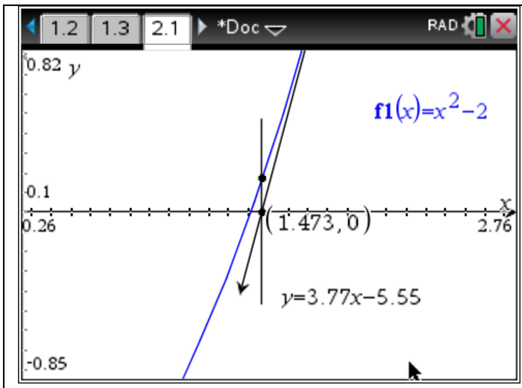
Now we can find where this tangent cuts the *x*-axis by solving the equation

$$6x - 11 = 0$$
$$x = \frac{11}{6}\,(1.83)$$

This is an improved estimate of the zero.

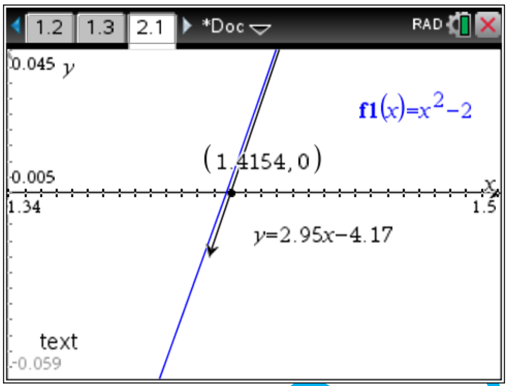
This method will generate improved approximations if we repeat it starting with drawing the tangent at $x = \frac{11}{6}$.

This time we will employ our TI Nspire CAS calculator.



Our next approximation is 1.473

Repeat the process again.



Our approximation now is 1.4154

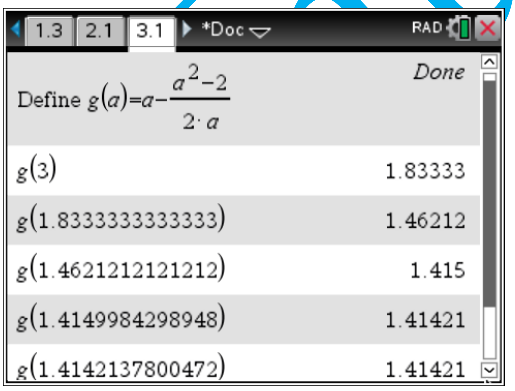
And so on.

Algebraically, we can use the iteration formula $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$.

So for the function above $f(x) = x^2 - 2$, starting at $x = 3$ gives us

$x = 3 - \dfrac{f(3)}{f'(4)}$ for the first approximation and $x_{n+1} = x_n - \dfrac{{x_n}^2}{2x_n}$.

This can be achieved in the following way on the calculator:



The bisection method videos:

https://www.youtube.com/watch?v=DAL6Hy97vn0
https://www.youtube.com/watch?v=swNe64ZBVj4
https://www.youtube.com/watch?v=TKecR7cizk8

Sample learning activity – linear equations for cubic roots

Introduction

This learning activity looks at using linear functions that are tangents to a graph to determine an approximate value for an irrational root of a cubic polynomial function. That is, the horizontal axis intercept of the tangent to the graph of the function at a point close to the root, approximates this root.

Part 1

Consider the cubic polynomial function $f : R \to R,\ f(x) = 2x^3 - 34x^2 - 196x - 373$

1. Plot the graph of the function and show that it only has one real root.
2. Find the unit interval [a, b] with integer endpoints that contains this root. By varying the plotting domain find the approximate value of the root correct to one decimal place.
3. Use the rational root theorem to show that this root is irrational.

Part 2

1. Find the horizontal axis intercept, b_1 of the tangent to the graph of f at $(b, f(b))$. Plot the graph of the tangent on the same set of axes as the graph of the function, with a suitable graphing domain to show the root and this approximation.
2. Find the horizontal axis intercept, b_2 of the tangent to the graph of f at $(b_1, f(b_1))$. Plot the graph of the tangent on the same set of axes as the graph of the function, with a suitable graphing domain to show the root and this second approximation.
3. Find the horizontal axis intercept, b_3 of the tangent to the graph of f at $(b_2, f(b_2))$. Plot the graph of the tangent on the same set of axes as the graph of the function, with a suitable graphing domain to show the root and this third approximation.

Part 3

1. Use technology to implement the algorithm for Newton's method, using both a and b as initial values.
2. Explore how quickly Newton's method works for other initial values.
3. Apply Newton's method to a cubic polynomial function with three real roots and explore the sensitivity of the methods to choice of initial value.

Apply the Newton Method to find approximate solutions to each of the following equations, correct to 3 decimal places:

(a) $\sin(2x) = x^2$

(b) The positive solution of $e^{2x} = 3\cos(x)$

(c) $x^3 = 3^x$

(d) $2e^{-x}\cos(4x) = 1$ in the interval [-1,0]

(e) $x^4 = x + 1$

Is the choice of the first estimate of the root significant? Explain your answer.